

COHEN–MACAULAY BINOMIAL EDGE IDEALS

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ABSTRACT. We study the depth of classes of binomial edge ideals and classify all closed graphs whose binomial edge ideal is Cohen–Macaulay

INTRODUCTION

Binomial edge ideals were introduced in [5]. They appear independently, and at about the same time, also in the paper [6]. In simple terms, a binomial edge ideal is just an ideal generated by an arbitrary collection of 2-minors of a $2 \times n$ -matrix whose entries are all indeterminates. Thus the generators of such an ideal are of the form $f_{ij} = x_i y_j - x_j y_i$ with $i < j$. It is then natural to associate with such an ideal the graph G on the vertex set $[n]$ for which $\{i, j\}$ is an edge if and only if f_{ij} belongs to our ideal. This explains the naming for this type of ideals. The binomial edge ideal of graph G is denoted by J_G . In [5] the relevance of this class of ideals for algebraic statistics is explained.

The goal of this paper is to characterize Cohen–Macaulay binomial edge ideals for simple graphs with vertex set $[n]$. Similar to ordinary edge ideals which were introduced by Villarreal [7], a general classification of Cohen–Macaulay binomial edge ideals seems to be hopeless. Thus we have to restrict our attention to special classes of graphs. In Section 1 we first consider the class of chordal graphs with the property that any two maximal cliques of it intersect in at most one vertex. These graphs include of course all forests. We show in Theorem 1.1 that for these graphs we have $\text{depth } S/J_G = n + c$, where n is the number of vertices of G and c is the number of connected components of G . As an application we show that the binomial edge ideal of a forest is Cohen–Macaulay if and only if each of its connected components is a path graph, and this is the case if and only if S/J_G is a complete intersection.

In Section 3 we use the results of Section 2 to give in Theorem 3.1 a complete characterization of all closed graphs whose binomial edge ideal is Cohen–Macaulay. Surprisingly this is the case if and only if its initial ideal is Cohen–Macaulay. Even more is true: if for a closed graph G , the ideal J_G is Cohen–Macaulay, then the graded Betti numbers of J_G and its initial ideal coincide. For a closed graph whose binomial edge ideal is Cohen–Macaulay, the Hilbert function and the multiplicity of S/J_G can be easily computed. Then by using the associativity formula of multiplicities in combination with the information given in [5] concerning the minimal prime ideals of binomial edge ideals we deduce in Corollary 3.6 certain numerical identities.

The term “closed graph” is not standard terminology in graph theory. It was introduced in [5] to characterize those graphs, which, for certain labeling of their edges, do have a quadratic Gröbner basis with respect to the lexicographic order induced by $x_1 > \cdots > x_n > y_1 > \cdots > y_n$. It is easy to see, as shown in [5], that any closed graph must be chordal. But by far not all chordal graphs are

closed. In Theorem 2.2 we give a description of the closed graphs which is then used in the proof of Theorem 3.1.

1. CLASSES OF CHORDAL GRAPHS WITH COHEN–MACAULAY BINOMIAL EDGE IDEAL

Recall that, by a result of Dirac [2] (see also [4]), a graph G is chordal if and only if it admits a *perfect elimination order*, that is, its vertices can be labeled $1, \dots, n$ such that for all $j \in [n]$, the set $C_j = \{i : i \leq j\}$ is a clique of G . A clique is simply a complete subgraph of G .

There is an equivalent characterization of chordal graphs in terms of its maximal cliques. To describe it we introduce some terminology. Let Δ be a simplicial complex. A facet F of Δ is called a *leaf*, if either F is the only facet, or else there exists a facet G , called a *branch* of F , which intersects F maximally. In other words, for each facet H of Δ with $H \neq F$ one has $H \cap F \subset G \cap F$. Each leaf F has at least one *free vertex*, that is, a vertex which belongs only to F . On the other hand, if a facet admits a free vertex it needs not to be a leaf.

The simplicial complex Δ is called a *quasi-forest* if its facets can be ordered F_1, \dots, F_r such that for all $i > 1$ the facet F_i is a leaf of the simplicial complex with facets F_1, \dots, F_{i-1} . Such an order of the facets is called a *leaf order*. A connected quasi-forest is called a *quasi-tree*.

Now let G be a graph. The collection of cliques of G forms a simplicial complex, called the *clique complex* of G . It is denoted $\Delta(G)$. The equivalent statement to Dirac's theorem now says that G is chordal if and only if $\Delta(G)$ is a quasi-forest.

In this section we will compute the depth of S/J_G for a very special class of chordal graphs. This class includes all forests. As a consequence it will be shown that a forest has a Cohen–Macaulay binomial edge ideal if and only if all its components are path graphs.

We shall need a few results from [5]. There in Corollary 3.9 and Corollary 3.3 the following fact is shown: Suppose that G is connected. Let $S \subset [n]$, and let $G_1, \dots, G_{c_G(S)}$ be the connected components of $G_{[n] \setminus S}$. For each G_i we denote by \tilde{G}_i the complete graph on the vertex set $V(G_i)$. If there is no confusion possible we simply write $c(S)$ for $c_G(S)$, and set

$$P_S(G) = \left(\bigcup_{i \in S} \{x_i, y_i\}, J_{\tilde{G}_1}, \dots, J_{\tilde{G}_{c(S)}} \right).$$

Then $J_G = \bigcap_{S \subset [n]} P_S(G)$, and $P_S(G)$ is a minimal prime ideal of J_G if and only if $S = \emptyset$, or $S \neq \emptyset$ and for each $i \in S$ one has $c(S \setminus \{i\}) < c(S)$. Moreover, $\text{height } P_S(G) = n + |S| - c(S)$ and hence $\dim S/J_G = \max\{(n - |S|) + c(S) : S \subset [n]\}$.

Theorem 1.1. *Let G be a chordal graph on $[n]$ with the property that any two distinct maximal cliques intersect in at most one vertex. Then $\text{depth } S/J_G = n + c$, where c is the number of connected components of G .*

Moreover, the following conditions are equivalent:

- (a) J_G is unmixed.
- (b) J_G is Cohen–Macaulay.
- (c) Each vertex of G is the intersection of at most two maximal cliques.

Proof. Let G_1, \dots, G_c be the connected components of G and set $S_i = K[\{x_j, y_j\}_{j \in G_i}]$. Then $S/J_G \cong S_1/J_{G_1} \otimes \dots \otimes S_c/J_{G_c}$, so that $\text{depth } S/J_G = \text{depth } S_1/J_{G_1} + \dots + \text{depth } S_c/J_{G_c}$. Thus in order to prove the desired result, we may assume that G is connected.

Let $\Delta(G)$ be the clique complex of G and let F_1, \dots, F_r be a leaf order on the facets of $\Delta(G)$. We make induction on r . If $r = 1$, then G is a simplex and the statement is true. Let $r > 1$; since F_r is a leaf, there exists a unique vertex, say $i \in F_r$, such that $F_r \cap F_j = \{i\}$ for some j . Let F_{t_1}, \dots, F_{t_q} be the facets of $\Delta(G)$ which intersect the leaf F_r in the vertex i .

Let $\mathcal{M}(G)$ denote the set of all sets $S \subset [n]$ such that $P_S(G)$ is a minimal prime ideal of J_G . We have $J_G = Q_1 \cap Q_2$ where $Q_1 = \bigcap_{S \in \mathcal{M}(G), i \notin S} P_S(G)$ and $Q_2 = \bigcap_{S \in \mathcal{M}(G), i \in S} P_S(G)$.

Consider the exact sequence

$$(1) \quad 0 \rightarrow S/J_G \rightarrow S/Q_1 \oplus S/Q_2 \rightarrow S/(Q_1 + Q_2) \rightarrow 0.$$

The ideal Q_1 is the binomial edge ideal associated with the graph G' which is obtained from G by replacing the facets F_{t_1}, \dots, F_{t_q} , and F_r by the clique on the vertex set $F_r \cup (\bigcup_{j=1}^q F_{t_j})$. Note that

G' is a connected chordal graph which has again the property that any two cliques intersect in at most one vertex, and it has a smaller number of cliques than G . Therefore, by induction, we have $\text{depth}(S/Q_1) = \text{depth}(S/J_{G'}) = n + 1$.

In order to determine Q_2 we first observe that for all $S \subset [n]$ with $i \in S$ we have that $P_S(G) = (x_i, y_i) + P_{S \setminus \{i\}}(G'')$, where G'' is the restriction of G to the vertex set $[n] \setminus \{i\}$. From this we conclude that $Q_2 = (x_i, y_i) + J_{G''}$. Let S_i be the polynomial ring $S/(x_i, y_i)$. Then $S/Q_2 \cong S_i/J_{G''}$. Hence, since G'' is a graph on $n - 1$ vertices and with $q + 1$ components satisfying the conditions of the theorem, our induction hypothesis implies that $\text{depth} S/Q_2 = (n - 1) + q + 1 = n + q$.

Next we observe that $Q_1 + Q_2 = J_{G'} + ((x_i, y_i) + J_{G''}) = (x_i, y_i) + J_{G'}$. Thus $S/(Q_1 + Q_2) \cong S_i/J_H$ where H is obtained from G' by replacing the clique on the vertex set $F_r \cup (\bigcup_{j=1}^q F_{t_j})$ by the clique on

the vertex set $F_r \cup (\bigcup_{j=1}^q F_{t_j}) \setminus \{i\}$. Thus our induction hypothesis implies that $\text{depth} S/(Q_1 + Q_2) = n$.

Hence the depth lemma applied to the exact sequence (1) yields the desired conclusion concerning the depth of S/J_G .

For the proof of the equivalence of statements (a), (b), and (c), we may again assume that G is connected. Let J_G be unmixed. Then $\dim(S/J_G) = n + 1$ since J_G has a minimal prime of dimension $n + 1$, namely $P_0(S)$. Since $\text{depth}(S/J_G) = n + 1$, it follows that J_G is Cohen-Macaulay, whence (a) \Rightarrow (b). The converse, (b) \Rightarrow (a), is well known.

(a) \Rightarrow (c): Let us assume that there is a vertex i of G where at least three cliques intersect. Then, for $S = \{i\}$, we get a minimal prime $P_S(G)$ of J_G of height strictly smaller than $n - 1$, which is in contradiction with the hypothesis on J_G .

(c) \Rightarrow (a): Let $\{i_1, \dots, i_{r-1}\}$ be the intersection vertices of the maximal cliques of G , and $P_S(G)$ a minimal prime of J_G . Let H_1, \dots, H_t be the connected components of $G_{[n] \setminus S}$. Suppose that there exists $i \in S \setminus \{i_1, \dots, i_{r-1}\}$. We have $c(S \setminus \{i\}) < c(S)$. This implies that there exists H_a, H_b , two connected components of $G_{[n] \setminus S}$, such that i is connected to H_a and H_b . Let $u \in V(H_a)$ and $v \in V(H_b)$ such that $\{i, u\}$ and $\{i, v\}$ are edges of G . Since $i \in S \setminus \{i_1, \dots, i_{r-1}\}$, it follows that u, v and i belong to the same clique of G , which implies that $\{u, v\}$ is an edge of G . Therefore, H_a and H_b are connected, a contradiction. By induction on the cardinality of S we see that $c(S) = |S| + 1$. Therefore, all the minimal primes of J_G have the same height. \square

As a consequence of Theorem 1.1 we obtain the following

Corollary 1.2. *Let G be a forest on the vertex set $[n]$. Then $\text{depth}(S/J_G) = n + c$, where c is the number of the connected components of G . Moreover, the following conditions are equivalent:*

- (a) J_G is unmixed;
- (b) J_G is Cohen-Macaulay;
- (c) J_G is a complete intersection;
- (d) Each component of G is a path graph.

Proof. The implications (c) \Rightarrow (b) \Rightarrow (a) are obvious, while (a) \Rightarrow (d) follows from Theorem 1.1. For the proof of (d) \Rightarrow (c) we may assume that G is a path, and the vertices are labeled in such a way such that $E(G) = \{\{i, i+1\} : i = 1, \dots, n-1\}$. Then $\text{in}_<(J_G) = (x_1y_2, x_2y_3, \dots, x_{n-1}y_n)$, where $<$ is the lexicographic order induced by $x_1 > x_2 > \dots > x_n > y_1 > y_2 > \dots > y_n$. Since the initial ideal of J_G is a complete intersection, J_G itself is a complete intersection. \square

The depth formula that we proved in Theorem 1.1 is not valid for arbitrary chordal graphs. For example for the graph G displayed in Figure 1 we have $\text{depth} S/J_G = 5$ (and not 6 as one would expect by Theorem 1.1). It is also an example of a graph for which J_G is unmixed but not Cohen–Macaulay.

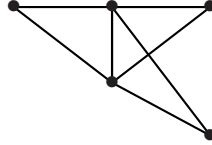


FIGURE 1.

2. CLOSED GRAPHS

In [5] the concept of closed graphs was introduced. In that paper a simple graph G on the vertex set $[n]$ is called *closed with respect to the given labeling*, if the following condition is satisfied:

- For all $\{i, j\}, \{k, l\} \in E(G)$ with $i < j$ and $k < l$ one has $\{j, l\} \in E(G)$ if $i = k$ but $j \neq l$, and $\{i, k\} \in E(G)$ if $j = l$ but $i \neq k$.

The definition was motivated by the following result [5, Theorem 1.1]: G is closed with respect to the given labeling, if and only if J_G has a quadratic Gröbner basis with respect to the lexicographic order induced by $x_1 > x_2 > \dots > x_n > y_1 > \dots > y_n$.

It is shown in [5, Proposition 1.4] that the graph G on $[n]$ is closed with respect to the given labeling, if and only if for any two integers $1 \leq i < j \leq n$ the shortest walk $\{i_1, i_2\}, \{i_2, i_3\}, \dots, \{i_{k-1}, i_k\}$ between i and j has the property that $i = i_1 < i_2 < \dots < i_k = j$. In particular, for each $i < n$ one has that $\{i, i+1\} \in E(G)$.

Definition 2.1. *We say a graph is closed if there exists a labeling for which it is closed.*

It arises the question to characterize the closed graphs. It is known from [5, Proposition 1.2] that if G is closed, then G is chordal.

Theorem 2.2. *Let G be a graphs on $[n]$. The following conditions are equivalent:*

- (a) G is closed;

(b) *there exists a labeling of G such that all facets of $\Delta(G)$ are intervals $[a, b] \subset [n]$.*

Moreover, if the equivalent conditions hold and the facets F_1, \dots, F_r of $\Delta(G)$ are labeled such that $\min(F_1) < \min(F_2) < \dots < \min(F_r)$, then F_1, \dots, F_r is a leaf order of $\Delta(G)$.

Proof. (a) \Rightarrow (b): Let G be a closed graph on $[n]$ and $F = \{j: \{j, n\} \in E(G)\}$, and let $k = \min\{j: j \in F\}$. Then $F = [k, n]$. Indeed, if $j \in F$ with $j < n$, then as observed above, it follows that $\{j, j+1\} \in E(G)$, and then because G is closed we see that since $\{j, n\} \in E(G)$, then also $\{j+1, n\} \in E(G)$. Thus $j+1 \in F$.

Next observe that F is a maximal clique of G , that is, a facet of $\Delta(G)$. First of all it is a clique, because $i, j \in F$ with $i < j < n$, then, since $\{i, n\}$ and $\{j, n\}$ are edges of G , it follows that $\{i, j\}$ is an edge as well, since G is closed. Secondly, it is maximal, since $\{j, n\} \notin E(G)$, if $j \notin F$.

Let $H \neq F$ be a facet of $\Delta(G)$ with $H \cap F \neq \emptyset$, and let $\ell = \max\{j: j \in H \cap F\}$. We claim that $H \cap F = [k, \ell]$. There is nothing to prove if $k = \ell$. So now suppose that $k < \ell$ and let $k \leq t < \ell$ and $s \in H \setminus F$. Then $s, t < \ell$ and $\{s, \ell\}$ and $\{t, \ell\}$ are edges of G . Hence since G is closed it follows that $\{s, t\} \in E(G)$. This implies that $s \in H$, as desired.

It follows from the claim that the facet H for which $\max\{j: j \in H \cap F\}$ is maximal, is a branch of F . In particular, F is a leaf. Let $H \cap F = [k, \ell]$, where H is a branch of F , and denote by G_ℓ the restriction of G to $[\ell]$. Since G_ℓ is again closed and since $\ell < n$, we may assume, by applying induction on the cardinality of the vertex set of G , that all facets of $\Delta(G_\ell)$ are intervals. Now let F' be any facet of $\Delta(G)$. If $F = F'$, then F is an interval, and if $F \neq F'$, then, as we have seen above, it follows that $F' \in \Delta(G')$. This yields the desired conclusion.

(b) \Rightarrow (a): Let $\{i, j\}$ and $\{k, \ell\}$ be edges of G with $i < j$ and $k < \ell$. If $i = k$, then $\{i, k\}$ and $\{i, \ell\}$ belong to the same maximal clique, that is, facet of $\Delta(G)$ which by assumption is an interval. Thus if $j \neq \ell$, then $\{j, \ell\} \in E(G)$. Similarly one shows that if $j = \ell$, but $i \neq k$, then $\{i, k\} \in E(G)$. Thus G is closed.

Finally it is obvious that the facets of $\Delta(G)$ ordered according to their minimal elements is a leaf order, because for this order F_{i-1} has maximal intersection with F_i for all i . \square

3. CLOSED GRAPHS WITH COHEN–MACAULAY BINOMIAL EDGE IDEAL

With the description of closed graphs given in Theorem 2.2 it is not hard to classify all closed graphs with Cohen–Macaulay binomial edge ideal.

Theorem 3.1. *Let G be a connected graph on $[n]$ which is closed with respect to the given labeling. Then the following conditions are equivalent:*

- (a) J_G is unmixed;
- (b) J_G is Cohen-Macaulay;
- (c) $\text{in}_<(J_G)$ is Cohen-Macaulay;
- (d) G satisfies the condition that whenever $\{i, j+1\}$ with $i < j$ and $\{j, k+1\}$ with $j < k$ are edges of G , then $\{i, k+1\}$ is an edge of G ;
- (e) *there exist integers $1 = a_1 < a_2 < \dots < a_r < a_{r+1} = n$ and a leaf order of the facets F_1, \dots, F_r of $\Delta(G)$ such that $F_i = [a_i, a_{i+1}]$ for all $i = 1, \dots, r$.*

Proof. We begin by proving (a) \Rightarrow (e). By Theorem 2.2, $\Delta(G)$ has facets F_1, \dots, F_r where each facet is an interval. We may order the intervals $F_i = [a_i, b_i]$ such that $1 = a_1 < a_2 < \dots < a_r \leq b_r = n$. Since G is connected it follows that $a_{i+1} \leq b_i$ for all i . Let $S = [a_r, b_{r-1}]$; then $c(S) = 2$, and so

height $P_S(G) = n + (b_{r-1} - a_r + 1) - 2 = n + (b_{r-1} - a_r) - 1$. On the other hand, height $P_\emptyset(G) = n - 1$, since G is connected. Thus our assumption implies that $n + (b_{r-1} - a_r) - 1 = n - 1$ which implies that $b_{r-1} = a_r$. Let G' be the graph whose clique complex $\Delta(G')$ has the facets F_1, \dots, F_{r-1} . Let $P_S(G')$ be a minimal prime ideal of G' . Then $b_{r-1} \notin S$. Therefore, $c_{G'}(S) = c_G(S)$, and hence $P_S(G)$ is a minimal prime ideal of J_G of same height as $P_S(G')$. Thus we conclude that $J_{G'}$ is unmixed as well. Induction on r concludes the proof.

In the sequence of implications (e) \Rightarrow (d) \Rightarrow (c) \Rightarrow (b) \Rightarrow (a), the second follows from the proof of [5, Proposition 1.6.], and the third and the fourth are well known for any ideal.

We prove (e) \Rightarrow (d). Let $i < j < k$ be three vertices of G such that $\{i, j+1\}$ and $\{j, k+1\}$ are edges of G . Then i and $j+1$ belong to the same facet of $\Delta(G)$, let us say to F_ℓ . Then $k+1$ must belong to F_ℓ as well since it is adjacent to j . Therefore, the condition from (d) follows. \square

Closed graphs with Cohen-Macaulay binomial edge ideal have the following nice property.

Proposition 3.2. *Let G be a closed graph with Cohen-Macaulay binomial edge ideal. Then $\beta_{ij}(J_G) = \beta_{ij}(\text{in}(J_G))$ for all i and j .*

Proof. For a graded S -module W we denote by $B_W(s, t) = \sum_{i,j} \beta_{ij}(W) s^i t^j$ the Betti polynomial of W .

Since $\text{in}(J_G)$ is Cohen-Macaulay, it follows from Theorem 3.1 that $[n] = \bigcup_{k=1}^r [a_k, a_{k+1}]$ with $1 = a_1 < a_2 < \dots < a_r < a_{r+1} = n$ and such that each $F_k := G_{[a_k, a_{k+1}]}$ is a clique. It follows that $\text{in}(J_G)$ is minimally generated by the set of monomials $\bigcup_{k=1}^r M_k$ where $M_k = \{x_i y_j : a_k \leq i < j \leq a_{k+1}\}$ for all k . Since for all $i \neq j$ the monomials of M_i and M_j are monomials in disjoint sets of variables, it follows that $\text{Tor}_k(S/(M_i), S/(M_j)) = 0$ for all $i \neq j$ and all $k > 0$. From this we conclude that

$$B_{S/\text{in}(J_G)}(s, t) = \prod_{i=1}^r B_{S/(M_i)}(s, t).$$

Since $\text{Tor}_k(S/(M_i), S/(M_j)) = 0$ for all $k > 0$, and since $\text{in}_<(J_{F_i}) = (M_i)$ for all i , we see that $\text{Tor}_k(S/J_{F_i}, S/J_{F_j}) = 0$ for all $k > 0$ as well. Thus we have

$$B_{S/J_G}(s, t) = \prod_{i=1}^r B_{S/J_{F_i}}(s, t).$$

Hence it remains to be shown that if G is a clique, then $\beta_{ij}(J_G) = \beta_{ij}(\text{in}(J_G))$ for all i and j . By the subsequent Lemma 3.3 and by Fröberg's theorem [3] we have that $\text{in}_<(J_G)$ has a 2-linear resolution. Therefore J_G has a 2-linear resolution as well. Thus for J_G and for $\text{in}_<(J_G)$, the Hilbert function of the ideal determines the Betti numbers. It is well-known that $S/\text{in}_<(J_G)$ and S/J_G have the same Hilbert function. Hence we conclude that the (graded) Betti numbers of J_G and $\text{in}_<(J_G)$ coincide. \square

Lemma 3.3. *Let G be a finite bipartite graph on $\{x_1, \dots, x_n\} \cup \{y_1, \dots, y_n\}$ with the edges $\{x_i, y_j\}$ with $1 \leq i \leq j \leq n$. Then the complementary graph \bar{G} of G is a chordal graph.*

Proof. Let $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_n\}$. Let C be a cycle of \bar{G} of length at least 5. Then C contains either three vertices belonging to X or three vertices belonging to Y . Since $\{x_i, x_j\}$ and $\{y_i, y_j\}$ are edges of \bar{G} for all $i \neq j$, it follows that C possesses a chord.

Now, let $C = (a, b, c, d)$ be a cycle of \bar{G} of length 4. If $a, c \in X$ or $b, d \in X$ or $a, c \in Y$ or $b, d \in Y$, then c possesses a chord. Suppose that $a \in X, c \in Y, b \in X$ and $d \in Y$, say, $C = (x_i, x_j, y_k, y_\ell)$. Then

$k < j$ and $\ell < i$. If $j < i$, then $k < i$. Thus $\{x_i, y_k\}$ is a chord of C . If $i < j$, then $\ell < j$. Thus $\{x_j, y_\ell\}$ is a chord of C . Hence \bar{G} is chordal, as desired. \square

Proposition 3.2 yields

Corollary 3.4. *Let G be a closed graph with Cohen–Macaulay binomial edge ideal, and assume that F_1, \dots, F_r are the facets of $\Delta(G)$ with $k_i = |F_i|$ for $i = 1, \dots, r$. Then the Cohen–Macaulay type of S/J_G is equal to $\prod_{i=1}^r (k_i - 1)$. In particular, S/J_G is Gorenstein if and only if G is a path graph.*

Proof. Due to Proposition 3.2 it suffices to show that if G is a clique on $[n]$ (with $n \geq 2$), then the Cohen–Macaulay type of S/J_G is equal to $n - 1$. In this particular case, J_G is the ideal of 2-minors of a $2 \times n$ -matrix whose resolution is given by the Eagon–Northcott complex. The type of S/J_G is the last Betti number in the resolution, which is $n - 1$. \square

Let G be a closed graph with Cohen–Macaulay binomial edge ideal, and assume that $F_1 = [a_1, a_2], \dots, F_r = [a_r, a_{r+1}]$, where $1 = a_1 < a_2 < \dots < a_r < a_{r+1} = n$, are the facets of $\Delta(G)$ and $k_i = |F_i|$ for $i = 1, \dots, r$. By using the well-known fact that S/J_G and $S/\text{in}(J_G)$ have the same Hilbert series, one easily gets the Hilbert series of S/J_G ,

$$H_{S/J_G}(t) = \frac{\prod_{i=1}^r [(k_i - 1)t + 1]}{(1 - t)^{n+1}}.$$

In particular, the multiplicity of S/J_G is $e(S/J_G) = k_1 \cdots k_r$ and the a -invariant is $a(S/J_G) = r - n - 1$.

By using the associativity formula for multiplicities we obtain a different expression for the multiplicity as the one given above. This will be a consequence of

Proposition 3.5. *$P_S(G)$ is a minimal prime of J_G if and only if S is empty or of the form $S = \{a_{j_1}, \dots, a_{j_s}\}$ for some $2 \leq j_1 < j_2 < \dots < j_s \leq r$ such that $a_{j_{q+1}} - a_{j_q} \geq 2$ for all $1 \leq q \leq s - 1$.*

In this case, the multiplicity of $S/P_S(G)$ is

$$e(S/P_S(G)) = (a_{j_1} - 1)(a_{j_2} - a_{j_1} - 1) \cdots (a_{j_s} - a_{j_{s-1}} - 1)(n - a_{j_s}).$$

Proof. For any s , if $S = \{a_{j_1}, \dots, a_{j_s}\}$ with $1 \leq j_1 < j_2 < \dots < j_s \leq r - 1$ such that $a_{j_{q+1}} - a_{j_q} \geq 2$ for all $1 \leq q \leq s - 1$, the number of the connected components of the restriction $G_{[n] \setminus S}$ of G is $s + 1$. This implies that for such S , $P_S(G)$ is a minimal prime ideal of J_G .

Conversely, let $S \neq \emptyset$, $S \subset [n]$, such that $P_S(G)$ is a minimal prime of G . In the first place we claim that S is contained in $\{a_2, \dots, a_r\}$. Indeed, let us suppose that there exists $j \in S \setminus \{a_2, \dots, a_r\}$, and let H_1, \dots, H_t be the connected components of $G_{[n] \setminus S}$. Since $P_S(G)$ is a minimal prime, we have $c(S \setminus \{j\}) < c(S)$. This implies that there exists some integers $a \neq b$ such that j is connected to H_a and H_b . Let $u \in V(H_a)$ and $v \in V(H_b)$ such that $\{u, j\}$ and $\{v, j\}$ are edges of G . Then u, v , and j belong to the same clique of G , thus $\{u, v\}$ is an edge of G and H_a, H_b are connected, which is impossible. Consequently, S is a subset of $\{a_2, \dots, a_r\}$. Let $S = \{a_{j_1}, \dots, a_{j_s}\}$ with $2 \leq j_1 < j_2 < \dots < j_s \leq r$ and assume that there exists $1 \leq q \leq s - 1$ such that $a_{j_{q+1}} = a_{j_q} + 1$. This means that $F_{j_q} = \{a_{j_q}, a_{j_q} + 1\}$. In this case it is easy to check that $c(S \setminus \{a_{j_q}\}) = c(S)$, which leads to a contradiction with the minimality of $P_S(G)$.

The formula for the multiplicity follows easily if we recall that the multiplicity of J_C is m if C is a clique with m vertices. \square

By comparing the two formulas for the multiplicity of S/J_G , we get the following

Corollary 3.6. *Let $b_1, \dots, b_r \geq 1$ be some integers. Then*

$$(b_1 + 1) \cdots (b_r + 1) = 1 + \sum_{i=1}^r b_i + \sum_{s=1}^{r-1} \sum_{1 \leq j_1 < \dots < j_s \leq r-1} \left[(b_1 + \dots + b_{j_1}) \prod_{i=1}^{s-1} (b_{j_i+1} + \dots + b_{j_{i+1}} - 1) (b_{j_s+1} + \dots + b_r) \right].$$

In particular, we have the following identity

$$2^r = \sum_{s=0}^{\lfloor \frac{r}{2} \rfloor} \sum_{(x_1, \dots, x_{s+1}) \in P(r-s+1|s+1)} x_1 \cdots x_{s+1},$$

where $P(r-s+1|s+1)$ stands for the set of all partitions of $r-s+1$ with $s+1$ parts.

In Proposition 3.2 we have seen that for a closed graph G , whose binomial edge ideal J_G is Cohen–Macaulay, the graded Betti numbers of J_G and $\text{in}_{<}(J_G)$ coincide. Computational evidence indicates that the graded Betti numbers of J_G and $\text{in}_{<}(J_G)$ coincide for all closed graphs. More generally, we conjecture that if G is a chordal graph whose clique complex $\Delta(G)$ has a leaf order F_1, \dots, F_r such that F_{i-1} is the unique branch of F_i for $i = 2, \dots, r$, then J_G and $\text{in}_{<}(J_G)$ have the same graded Betti numbers.

We call chordal graphs with the above property on the leaf order *chain of cliques*. Each closed graphs is a chain of cliques as we have seen in Theorem 2.2. The converse is not true, as the following example shows:

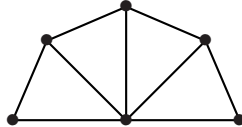


FIGURE 2.

Based on explicit calculations and general arguments in special cases we believe that in general for all graphs G the extremal Betti numbers (see [1]) of J_G and $\text{in}_{<}(J_G)$ coincide.

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